Lower-Bounding the MTTF for systems with (m,k) constraints and IID iteration failure probabilities

Arpan Gujarati,
Mitra Nasri, Björn B. Brandenburg

MAX PLANCK INSTITUTE
FOR SOFTWARE SYSTEMS

Mean Time To Failure

Lower-Bounding the MTTF for systems with (m,k) constraints and IID iteration failure probabilities

Independent and Identically Distributed

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Reliability analysis of Networked Control Systems (NCS)

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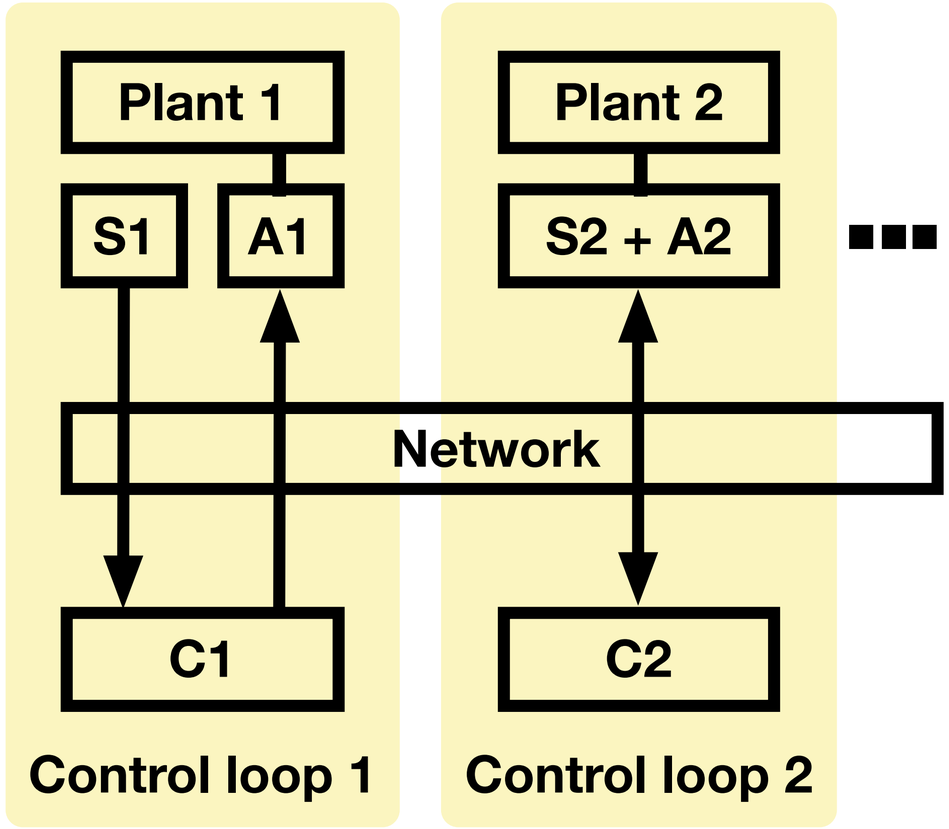
= multiple feedback control loops + distributed hosts
+ shared communication network



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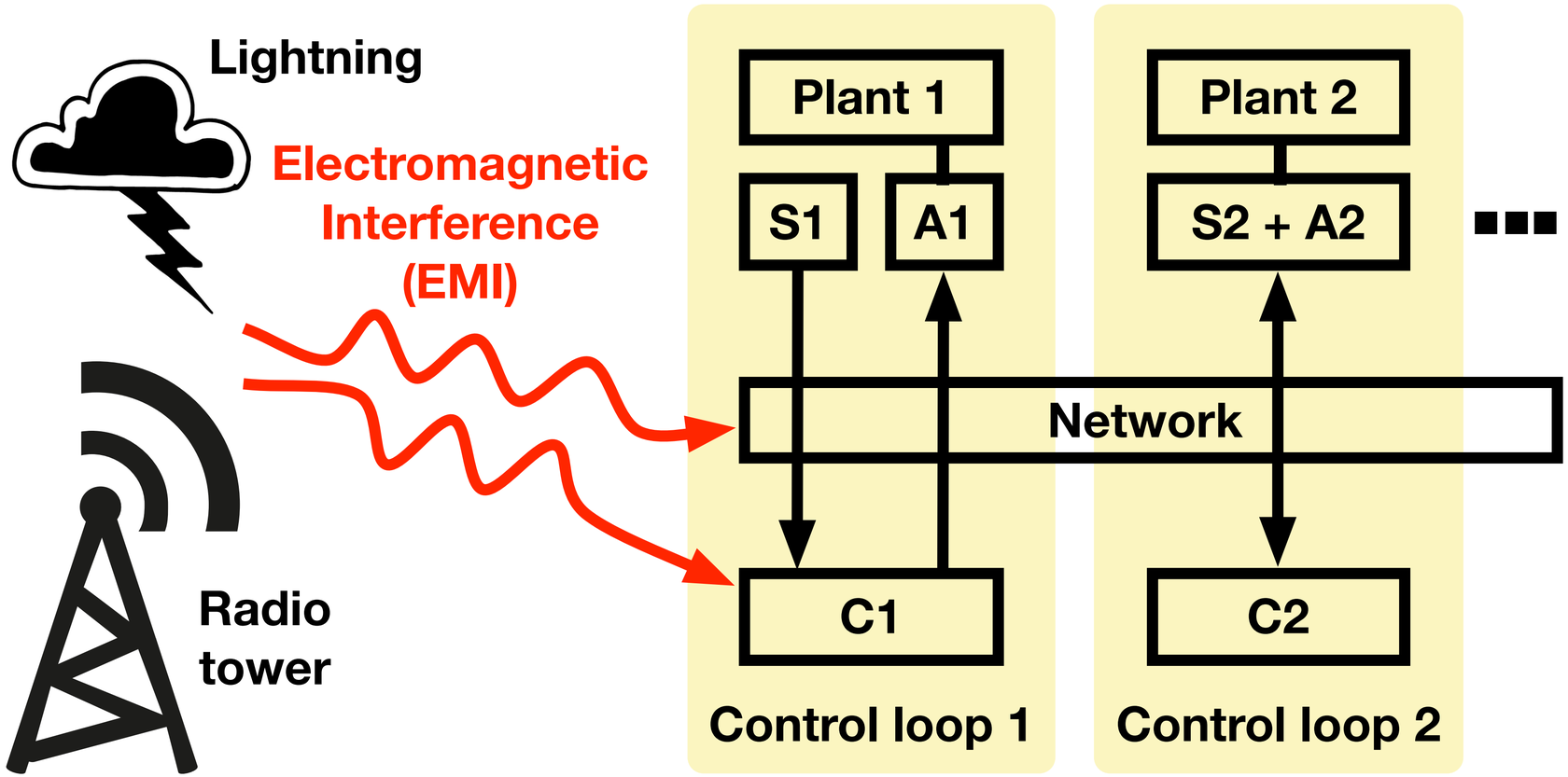

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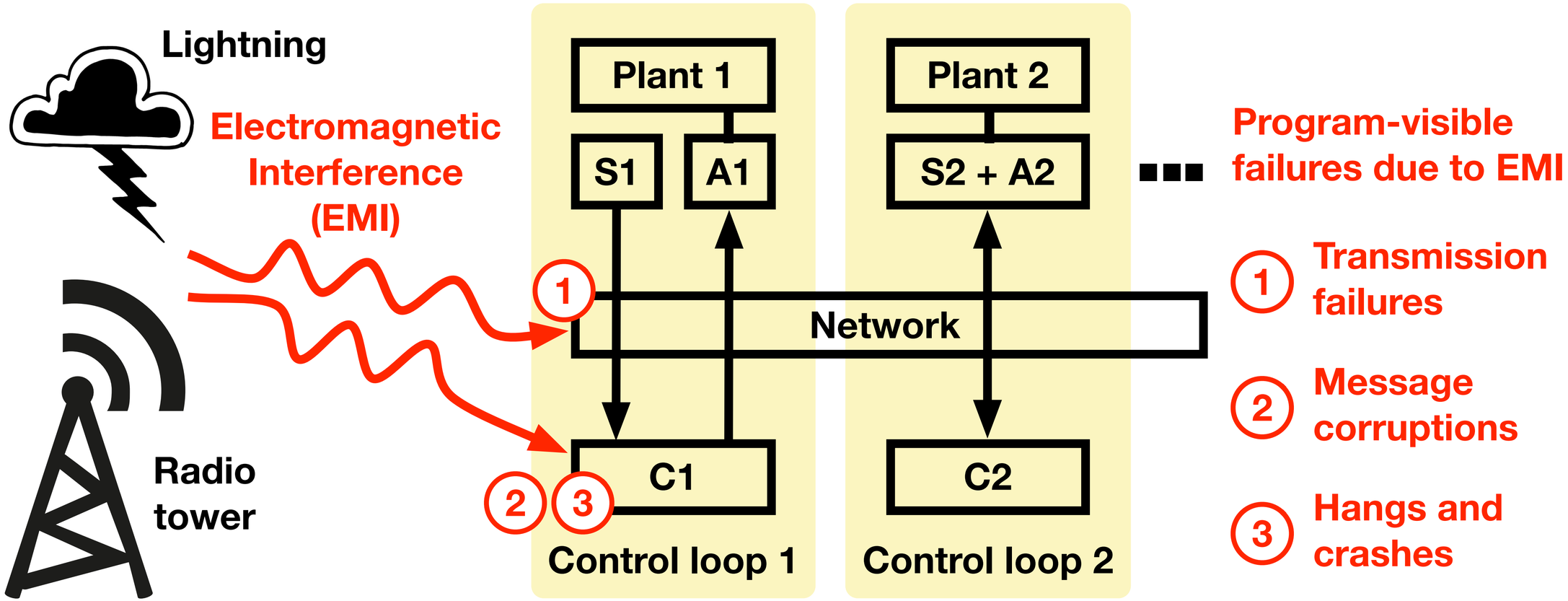

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i.e., continue functioning despite EMI-induced failures

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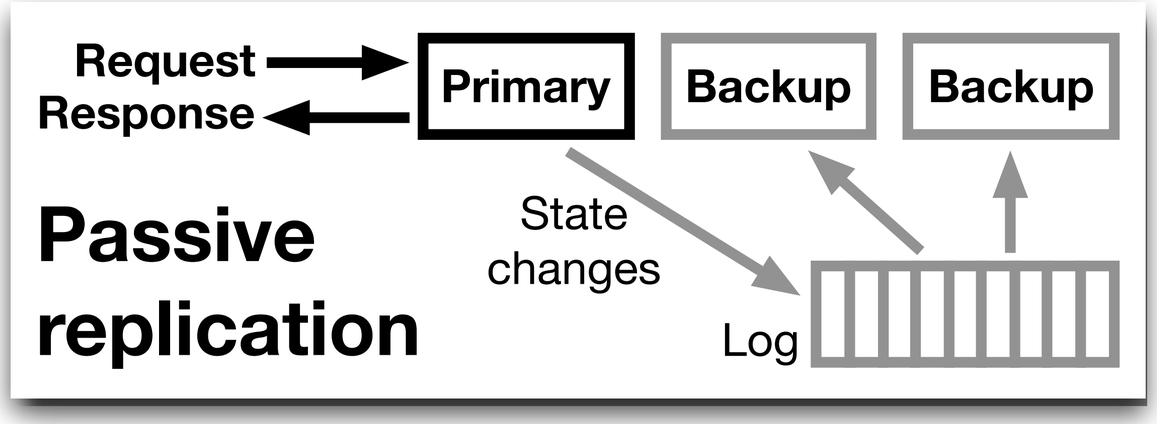

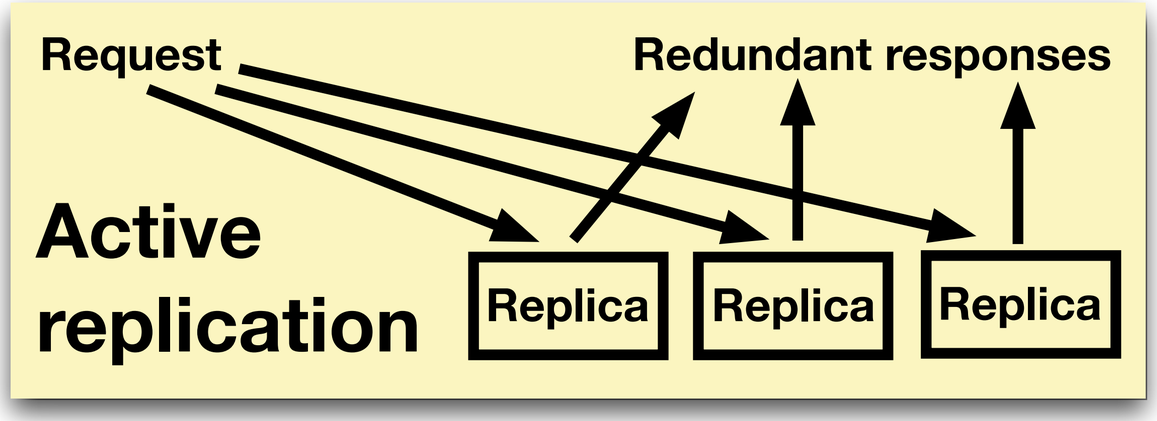

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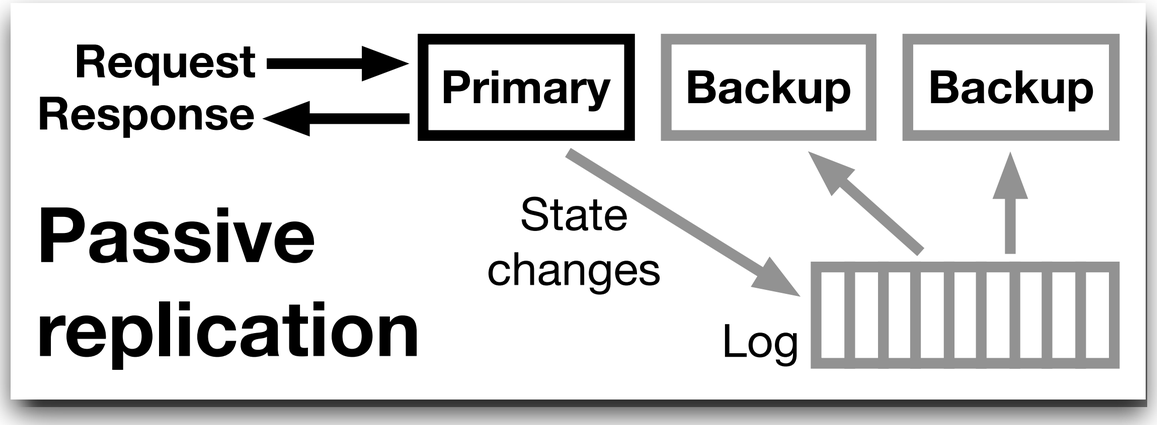

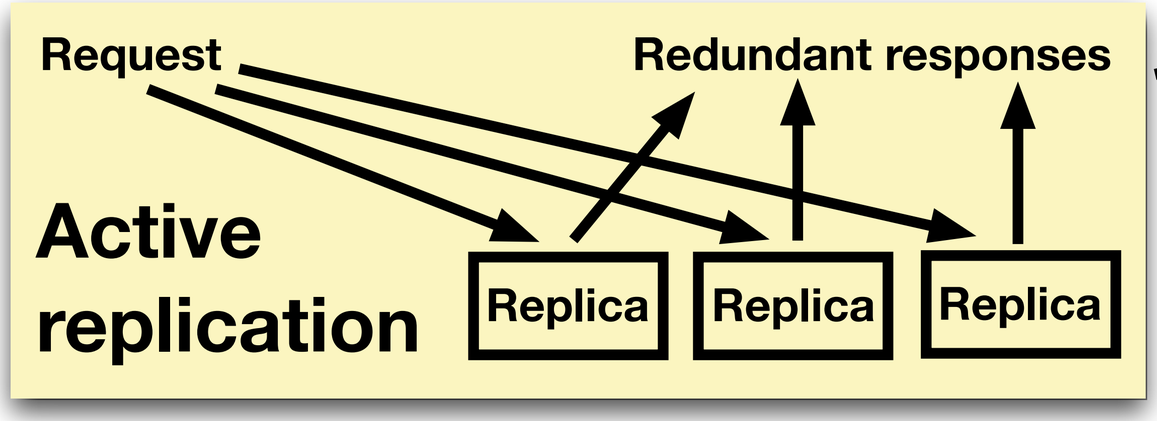

Active replication is often used because…

A. NCSs are time-sensitive

B. they contain high-frequency control loops

Problem

What is a good active replication scheme?

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Constraints: size, weight, power, and cost

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Opportunity: controller inherently robust to occasional disturbances

Quantifying NCS resiliency to EMI-induced transient faults

… to provide engineers with an objective metric for comparing different active replication schemes

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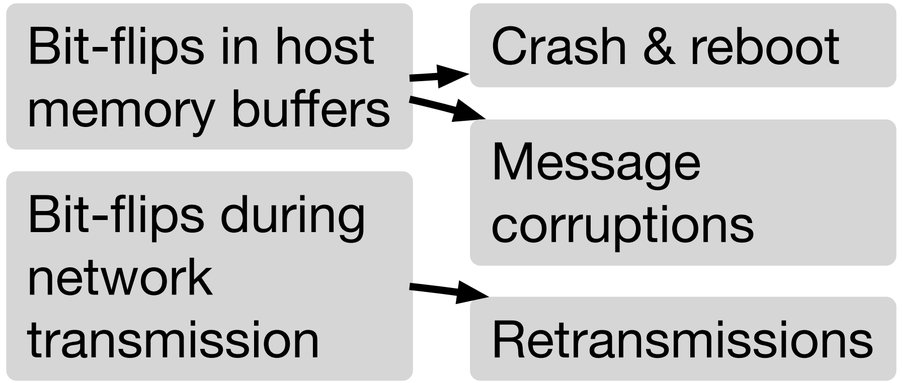

(1) Fault Modeling: Transient faults modeled using **Poisson** distribution, empirically-derived **peak EMI** rates

Quantifying NCS resiliency to EMI-induced transient faults

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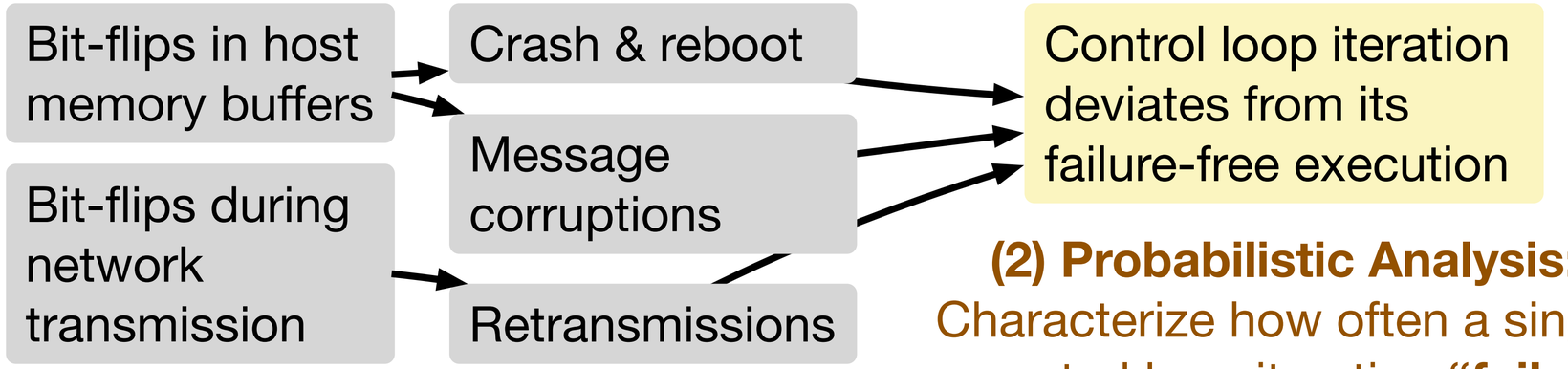

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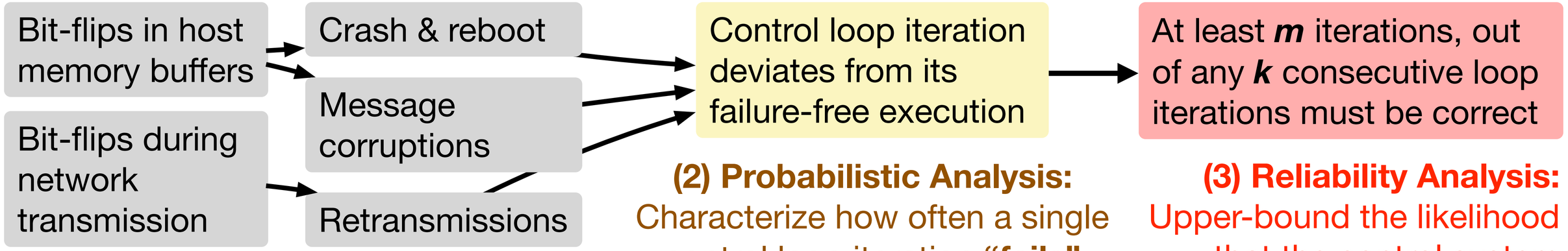

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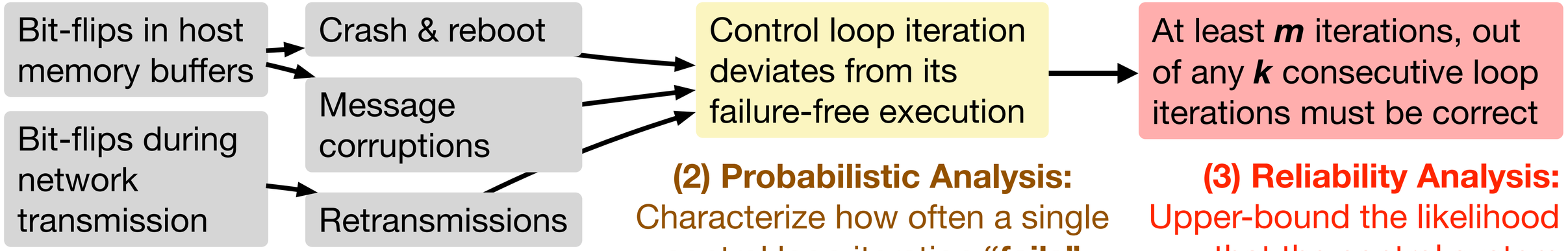

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This probability is upper-bounded by **F**, which satisfies the **IID** property w.r.t. each iteration

(under submission)

Quantifying NCS resiliency to EMI-induced transient faults

… to provide engineers with an objective metric for comparing different active replication schemes

At least ***m*** iterations, out of any ***k*** consecutive loop iterations must be correct

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Given F, lower-bound the Mean Time To Failure (MTTF)

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violation of the *(m,k)* constraint

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Outline

Failure = Violation of the (m,k) constraint:

At least ***m*** iterations, out of any ***k*** consecutive loop iterations must be correct

Given F, lower-bound the mean time to failure (MTTF)

Outline

1. Discrete probability density function (dPDF)
   g(n) = P(first (m,k) violation in the n^{th} iteration)

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Lower-bounding dPDF (1/3)

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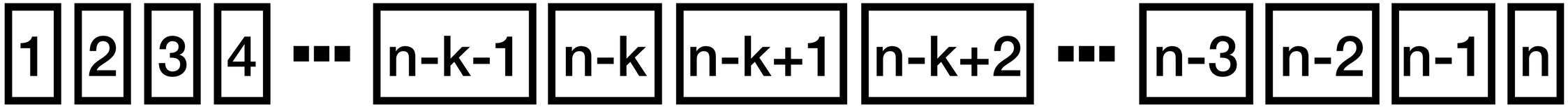

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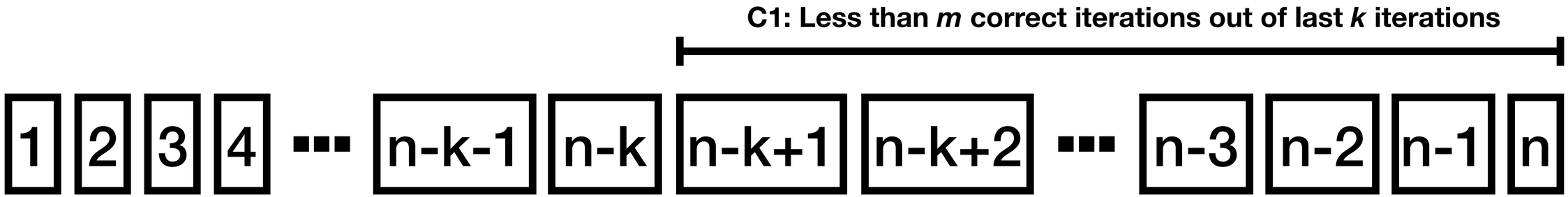

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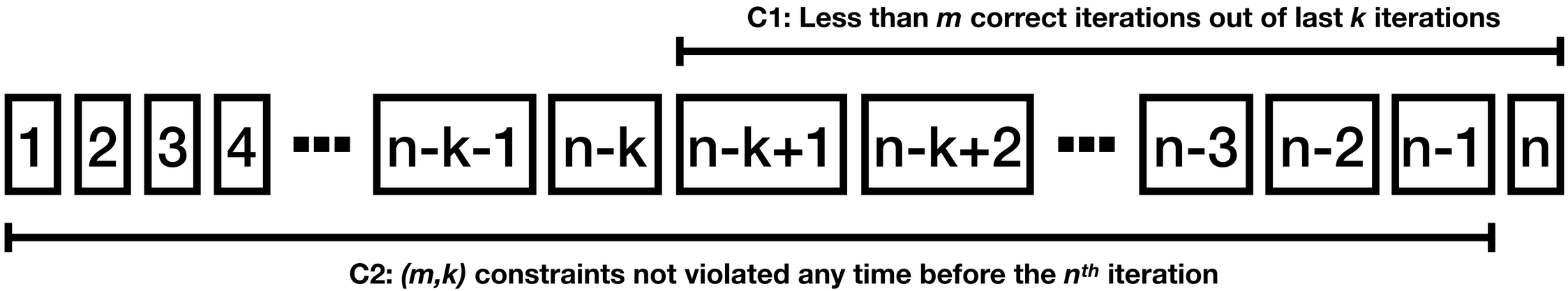

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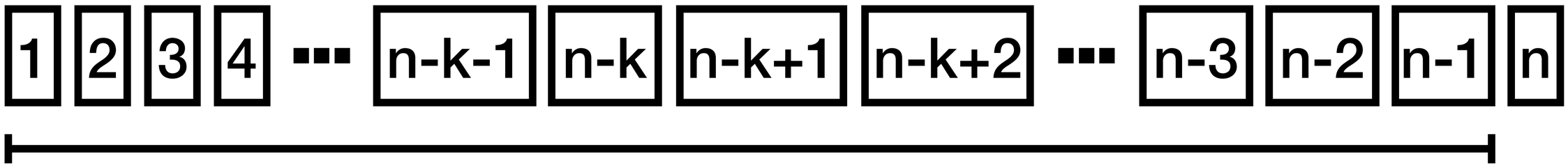
$$P(C1) = \binom{k-1}{k-m} F^{(k-m+1)}(1-F)^{m-1}$$



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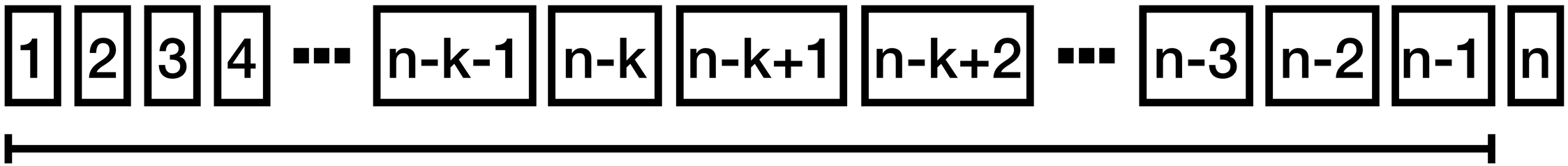

$$P(C2) = ?$$

Lower-bounding dPDF (1/3)

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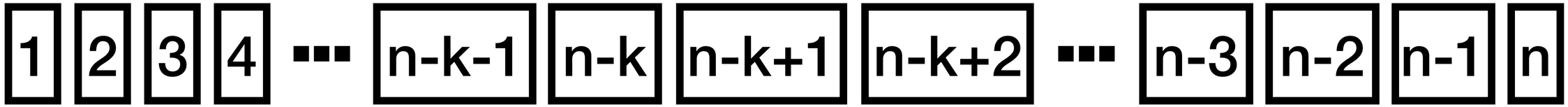
C1: Less than *m* correct iterations out of last *k* iterations



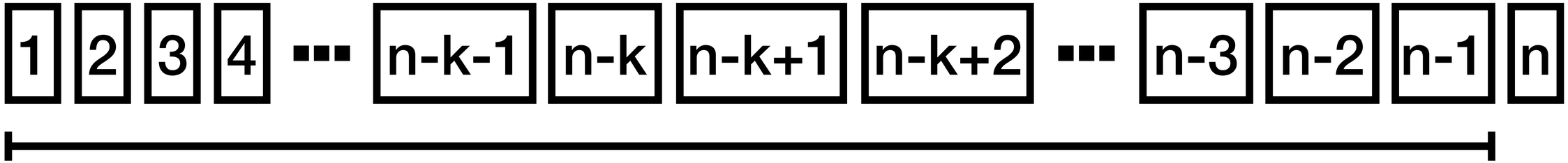
C2: *(m,k)* constraints not violated any time before the n^{th} iteration

Computationally challenging

$$P(C2) = ?$$

Requires evaluating all possible combinations of failed and successful iterations among the first *n – 1* iterations.

Lower-bounding dPDF (2/3)



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modeled as

a-within-consecutive-b-out-of-c:F system

- consists of c (c ≥ a) linearly ordered components,
- fails iff at least a (a ≤ b) components fail among any b consecutive components.

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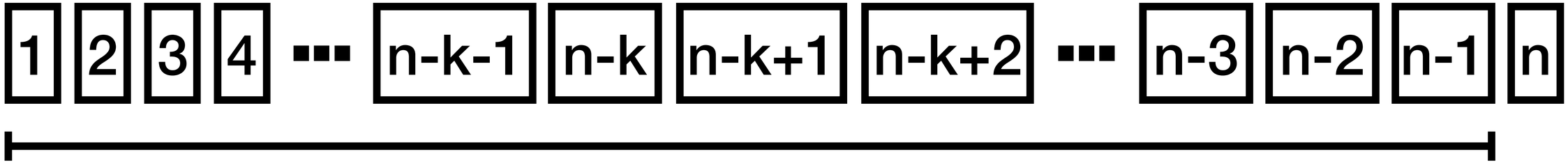

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Sfakianakis et al. (1992)

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$$P(C2) >= R_{abc}(k - m + 1,\ k,\ n - 1)$$

Lower-bounding dPDF (3/3)

$$P(C1) = \binom{k-1}{k-m} F^{(k-m+1)} (1-F)^{m-1}$$

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$$P(C1) = \binom{k-1}{k-m} F^{(k-m+1)}(1-F)^{m-1}$$

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$$g(n) \geq g_{LB}(n) = \binom{k-1}{k-m} F^{(k-m+1)}(1-F)^{m-1} R_{abc}(k-m+1,\ k,\ n-1)$$

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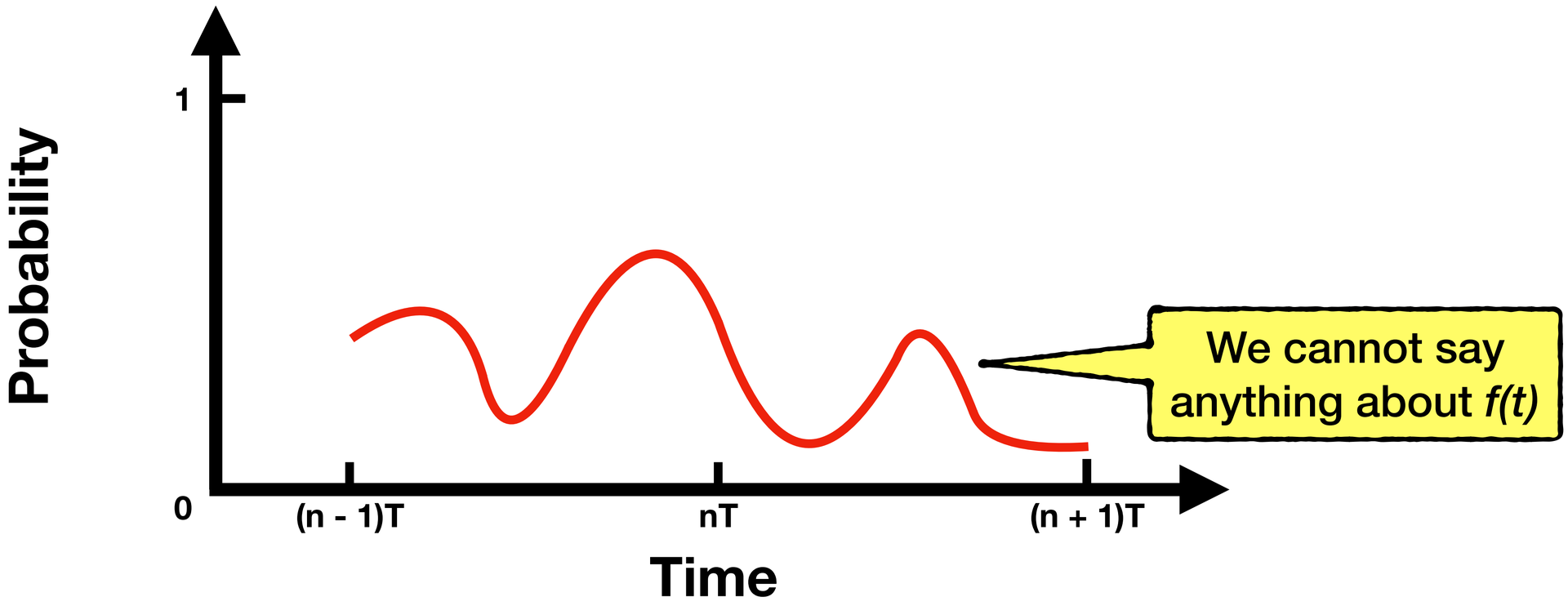
Lower-bounding PDF using dPDF lower bound

$f(t)$

$g_{LB}(n)$

Lower-bounding PDF using dPDF lower bound

f(t) *$g_{LB}(n)$*



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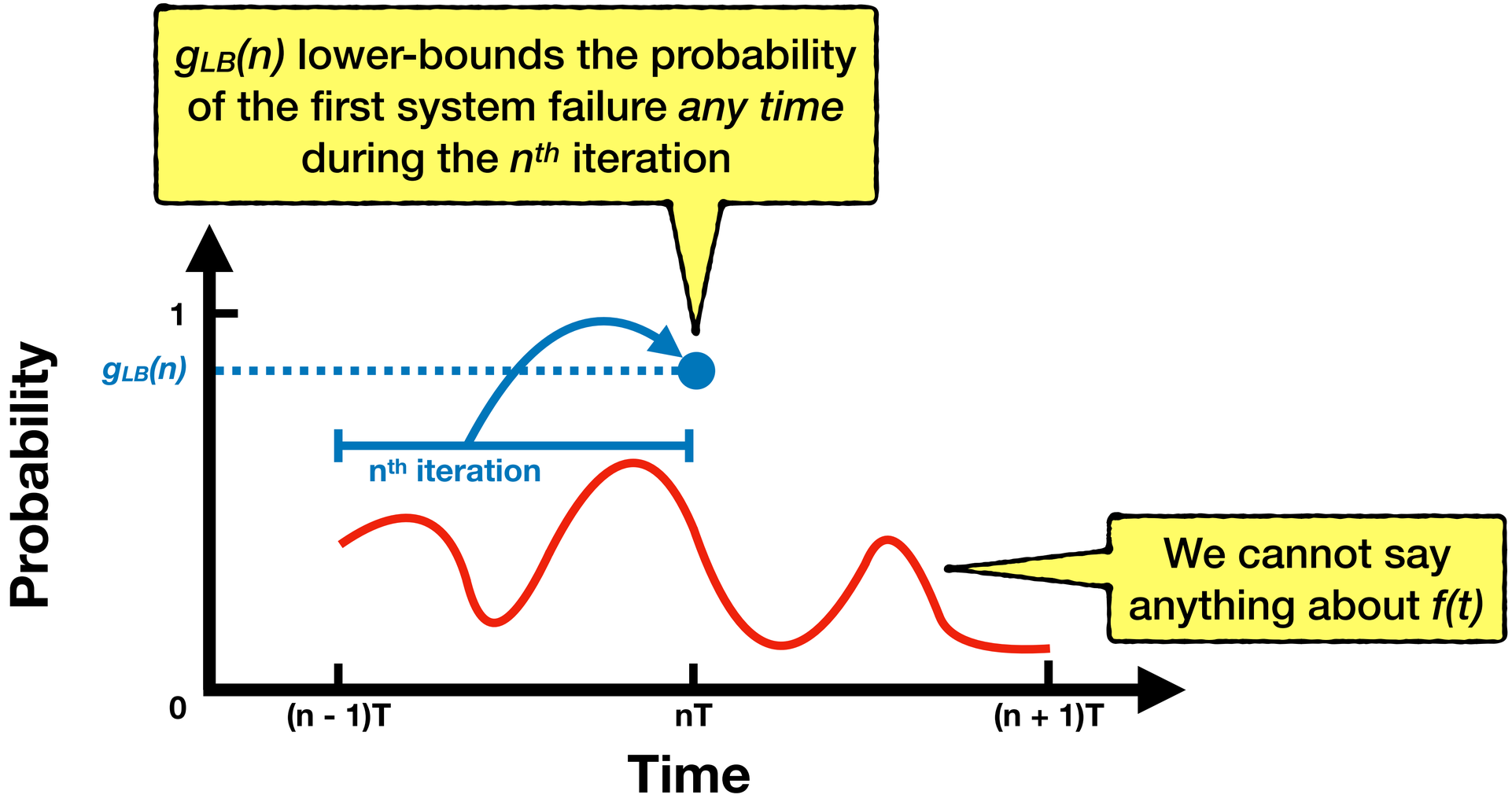

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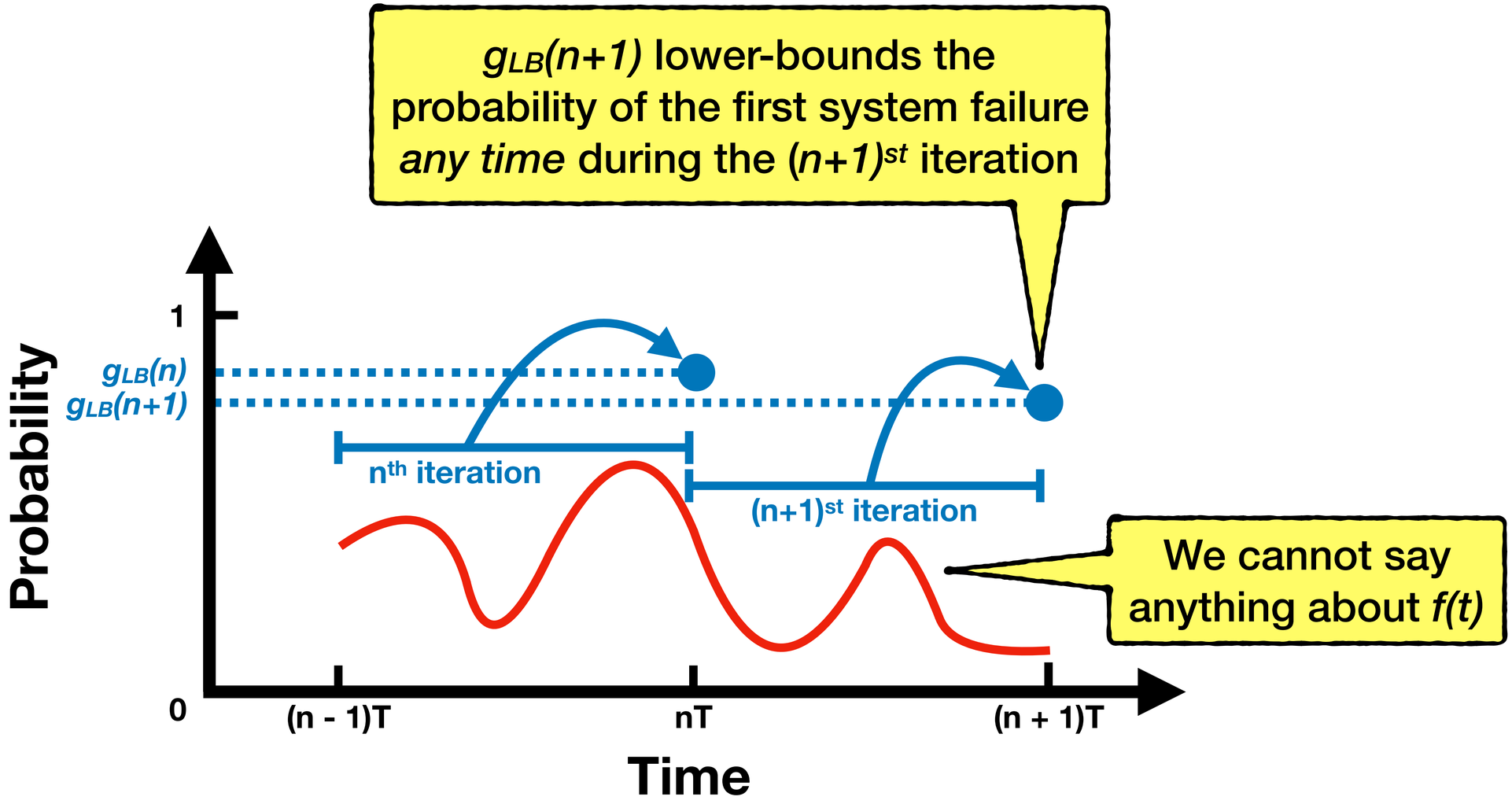


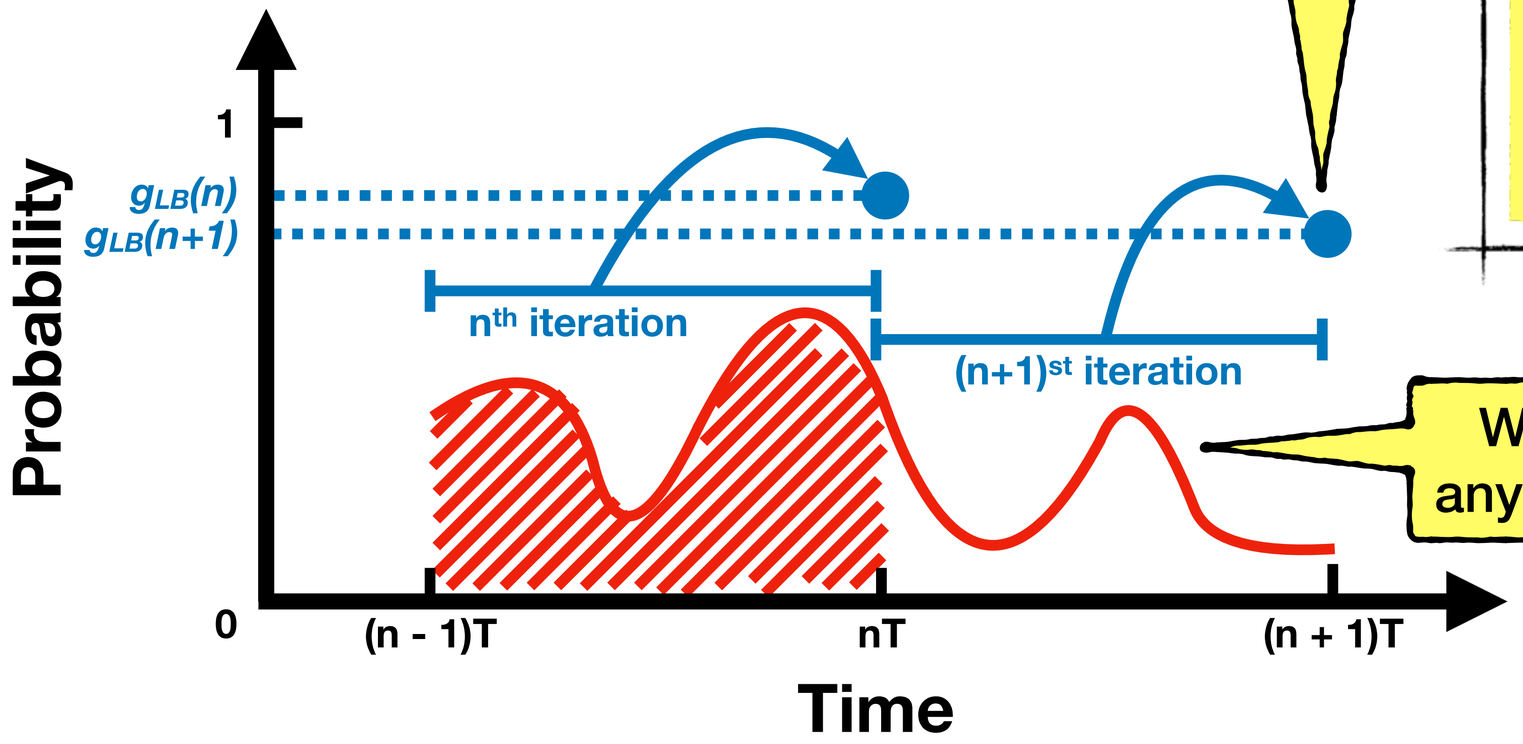
Lower-bounding PDF using dPDF lower bound

f(t)

$g_{LB}(n)$

$g_{LB}(n+1)$ lower-bounds the probability of the first system failure *any time* during the *(n+1)st* iteration

$$\int_{(n-1)T}^{nT} f(t) \geq g_{LB}(n)$$



We cannot say anything about *f(t)*

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Challenges

#	Case	Definition	Type	Source
1	$a = 0$	$R_1(a,b,c) = 0$	Exact	–
2	$a = 1$	$R_2(a,b,c) = P_S^c$	Exact	–
3	$a = 2 \wedge c \leq 4b$	$R_3(a,b,c) = \sum_{i=0}^{\lfloor \frac{c+b-1}{b} \rfloor} \binom{c-(i-1)(b-1)}{i} P_F^i P_S^{c-i}$	Exact	[12, §11.4.1] (Eqs. 11.9 and 11.10)
4	$a = 2 \wedge c > 4b$	$R_4(a,b,c) = R_3(a,b,b+t-1)(R_3(a,b,b+3))^u$ where $t = (c-b+1) \bmod 4$ and $u = \lfloor \frac{c-b+1}{4} \rfloor$	LB	[12, §11.4.1] (Eq. 11.16)
5	$a > 2 \wedge c \leq 2b \wedge a = b$	$R_5(a,b,c) = \begin{cases} 1 & 0 \leq c < a \\ 1 - P_F^a - (c-k)P_F^a P_S & a \leq c \leq 2a \end{cases}$	Exact	[12, §9.1.1] (Eqs. 9.2, 9.9, and 9.20)
6	$a > 2 \wedge c \leq 2b \wedge a \neq b \wedge c \leq b$	$R_6(a,b,c) = \sum_{i=c-a+1}^{c} \binom{c}{i} P_S^i P_F^{c-i}$	Exact	[12, §7.1.1] (Eq. 7.2)
7	$a > 2 \wedge c \leq 2b \wedge a \neq b \wedge c > b$	$R_7(a,b,c) = \sum_{i=0}^{a-1} \binom{b-s}{i} P_F^i P_S^{b-s-i} M(a',s,2s)$ where $s = c-b$ and $a' = a-i$, and $M(a',s,2s) = \begin{cases} 1 & a' > s \\ R_2(a',s,2s) & a' = 1 \\ R_3(a',s,2s) & a' = 2 \\ R_5(a',s,2s) & a' > 2 \wedge a' = s \\ R_7(a',s,2s) & a' > 2 \wedge a' \neq s \end{cases}$	Exact	[12, §11.4.1] (Eq. 11.14)
8	$a > 2 \wedge c > 2b$	$R_8(a,b,c) = R_\phi(a,b,b+t-1)(R_\phi(a,b,b+3))^u$ where $t = (c-b+1) \bmod 4$ and $u = \lfloor \frac{c-b+1}{4} \rfloor$, and $R_\phi(a,b,c) = \begin{cases} R_5(a,b,c) & a = b \\ R_6(a,b,c) & a \neq b \wedge a \leq b \\ R_7(a,b,c) & a \neq b \wedge a > b \end{cases}$	LB	[12, §11.4.1] (Eq. 11.16)

TABLE I. **Type** indicates whether the reliability definition for that respective case is an exact value or a lower bound.

$$^{1}R_{abc}(k-m+1,\ k,\ n-1)$$

Problem

- Complex definition
- Multiple sub-cases
- Recursive expressions

Challenges

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1	$a = 0$	$R_1(a,b,c) = 0$	Exact	–
2	$a = 1$	$R_2(a,b,c) = P_S^c$	Exact	–
3	$a = 2 \wedge c \leq 4b$	$R_3(a,b,c) = \sum_{i=0}^{\lfloor \frac{c+b-1}{b} \rfloor} \binom{c-(i-1)(b-1)}{i} P_F^i P_S^{c-i}$	Exact	[12, §11.4.1] (Eqs. 11.9 and 11.10)
4	$a = 2 \wedge c > 4b$	$R_4(a,b,c) = R_3(a,b,b+t-1)(R_3(a,b,b+3))^u$ where $t = (c-b+1) \bmod 4$ and $u = \lfloor \frac{c-b+1}{4} \rfloor$	LB	[12, §11.4.1] (Eq. 11.16)
5	$a > 2 \wedge c \leq 2b \wedge a = b$	$R_5(a,b,c) = \begin{cases} 1 & 0 \leq c < a \\ 1 - P_F^a - (c-k)P_F^a P_S & a \leq c \leq 2a \end{cases}$	Exact	[12, §9.1.1] (Eqs. 9.2, 9.9, and 9.20)
6	$a > 2 \wedge c \leq 2b \wedge a \neq b \wedge c \leq b$	$R_6(a,b,c) = \sum_{i=c-a+1}^{c} \binom{c}{i} P_S^i P_F^{c-i}$	Exact	[12, §7.1.1] (Eq. 7.2)
7	$a > 2 \wedge c \leq 2b \wedge a \neq b \wedge c > b$	$R_7(a,b,c) = \sum_{i=0}^{a-1} \binom{b-s}{i} P_F^i P_S^{b-s-i} M(a',s,2s)$ where $s = c-b$ and $a' = a-i$, and $M(a',s,2s) = \begin{cases} 1 & a' > s \\ R_2(a',s,2s) & a' = 1 \\ R_3(a',s,2s) & a' = 2 \\ R_5(a',s,2s) & a' > 2 \wedge a' = s \\ R_7(a',s,2s) & a' > 2 \wedge a' \neq s \end{cases}$	Exact	[12, §11.4.1] (Eq. 11.14)
8	$a > 2 \wedge c > 2b$	$R_8(a,b,c) = R_\phi(a,b,b+t-1)(R_\phi(a,b,b+3))^u$ where $t = (c-b+1) \bmod 4$ and $u = \lfloor \frac{c-b+1}{4} \rfloor$, and $R_\phi(a,b,c) = \begin{cases} R_5(a,b,c) & a = b \\ R_6(a,b,c) & a \neq b \wedge a \leq b \\ R_7(a,b,c) & a \neq b \wedge a > b \end{cases}$	LB	[12, §11.4.1] (Eq. 11.16)

TABLE I. **Type** indicates whether the reliability definition for that respective case is an exact value or a lower bound.

$$R_{abc}(k-m+1,\ k,\ n-1)$$

Problem

- Complex definition
- Multiple sub-cases
- Recursive expressions

Symbolic integration not an option!

Numeric, but sound, method to lower-bound the MTTF

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$$g(n) \geq \underbrace{g_{LB}(n)} = \binom{k-1}{k-m} F^{(k-m+1)} (1-F)^{m-1} R_{abc}(k-m+1,\ k,\ n-1)$$

Computing $g_{LB}(n)$ for a given < *m, k, n, F* > is easy

- ***m, k, F* are constants for a given system**

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- ***m, k, F*** **are constants for a given system**

But what about *n*?

- **n varies from 0 to ∞**

$$\int_{(n-1)T}^{nT} f(t) \geq g_{LB}(n)$$

$$MTTF = \int_0^\infty t f(t)\, dt$$

Compute $g_{LB}(n)$ at L + 1 distinct points $d_0, d_1, ..., d_L$

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Paper

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using the relation between PDF and dPDF

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Choosing points $d_0, d_1, \ldots, d_L$

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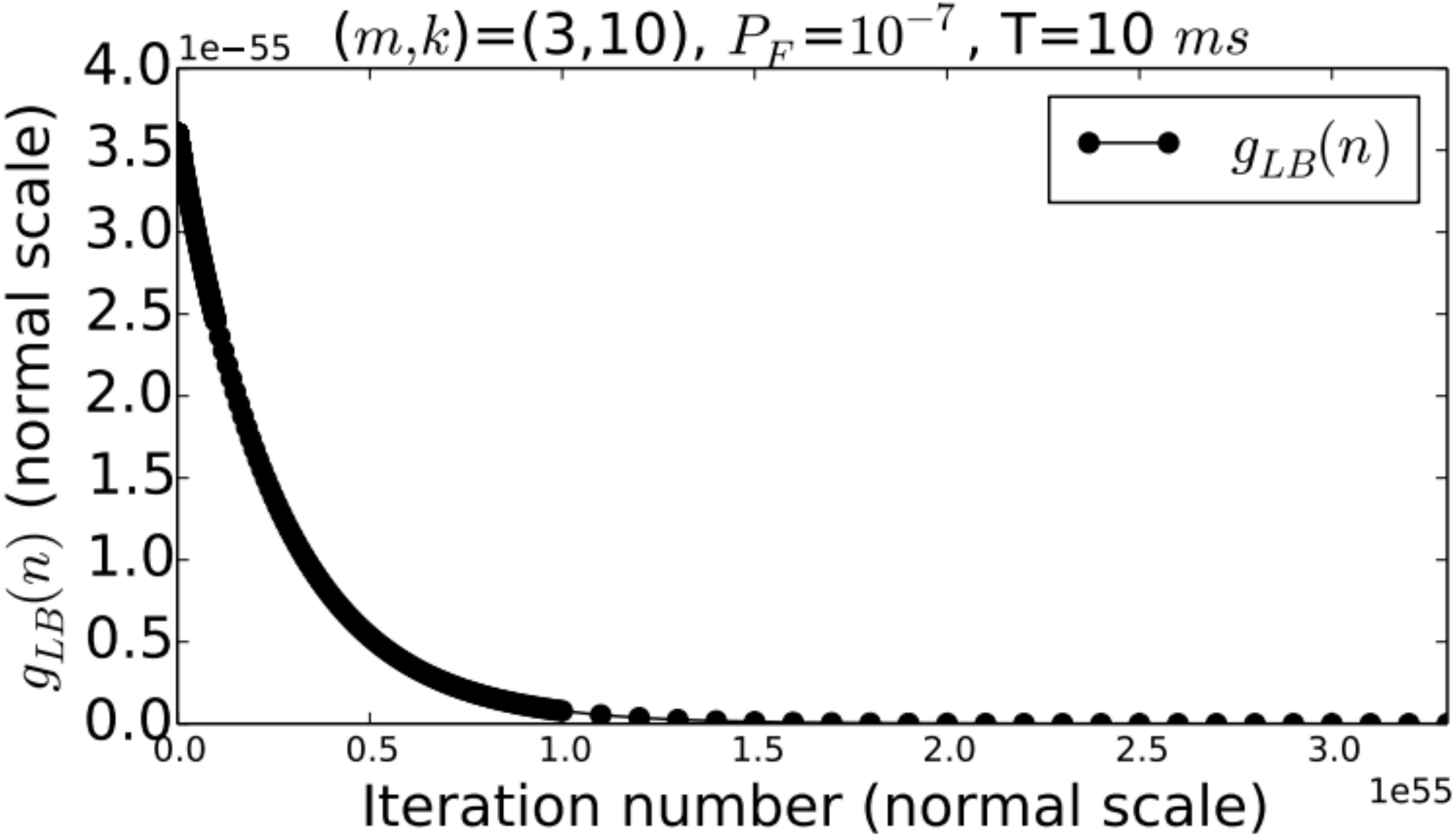

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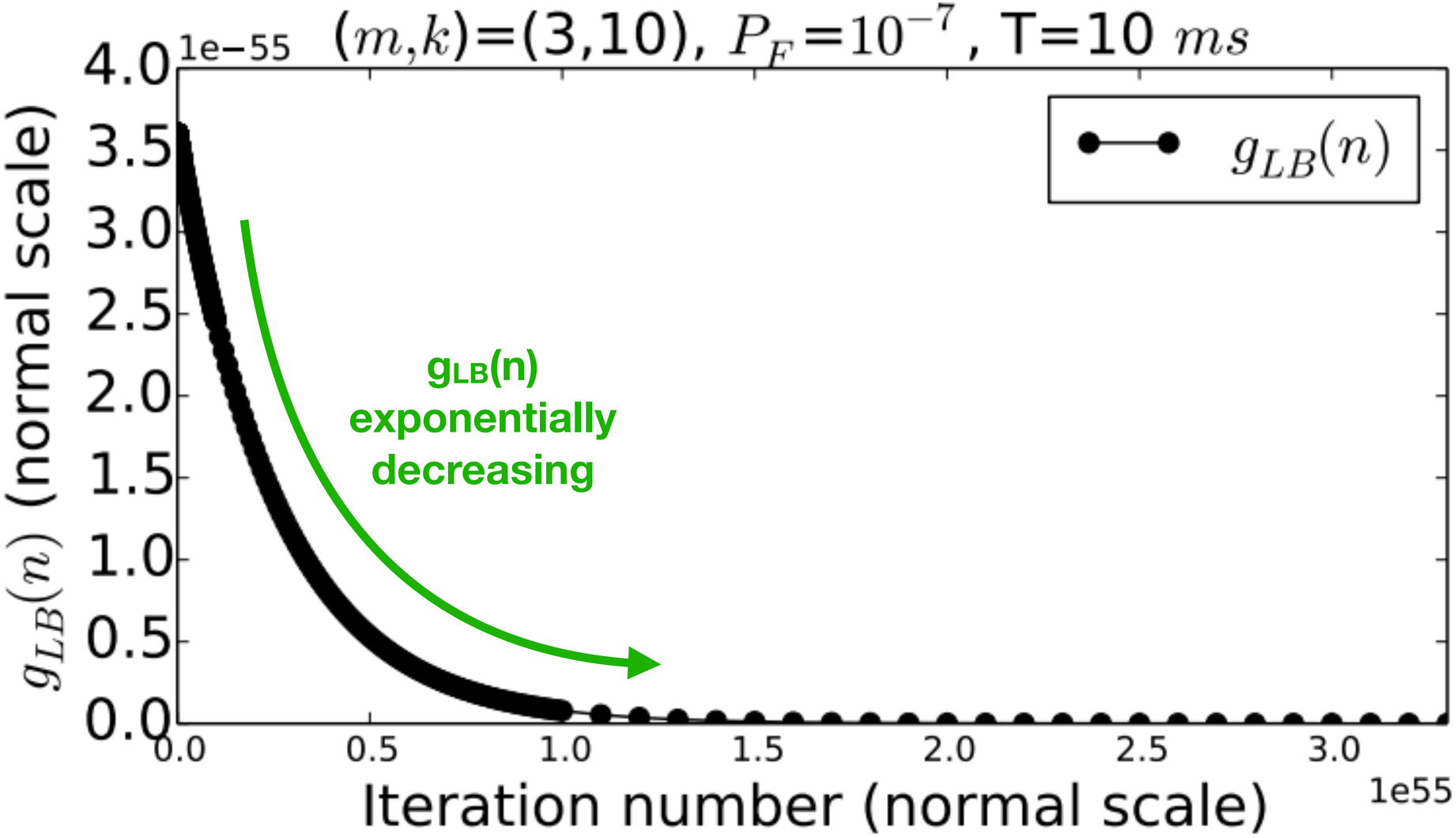

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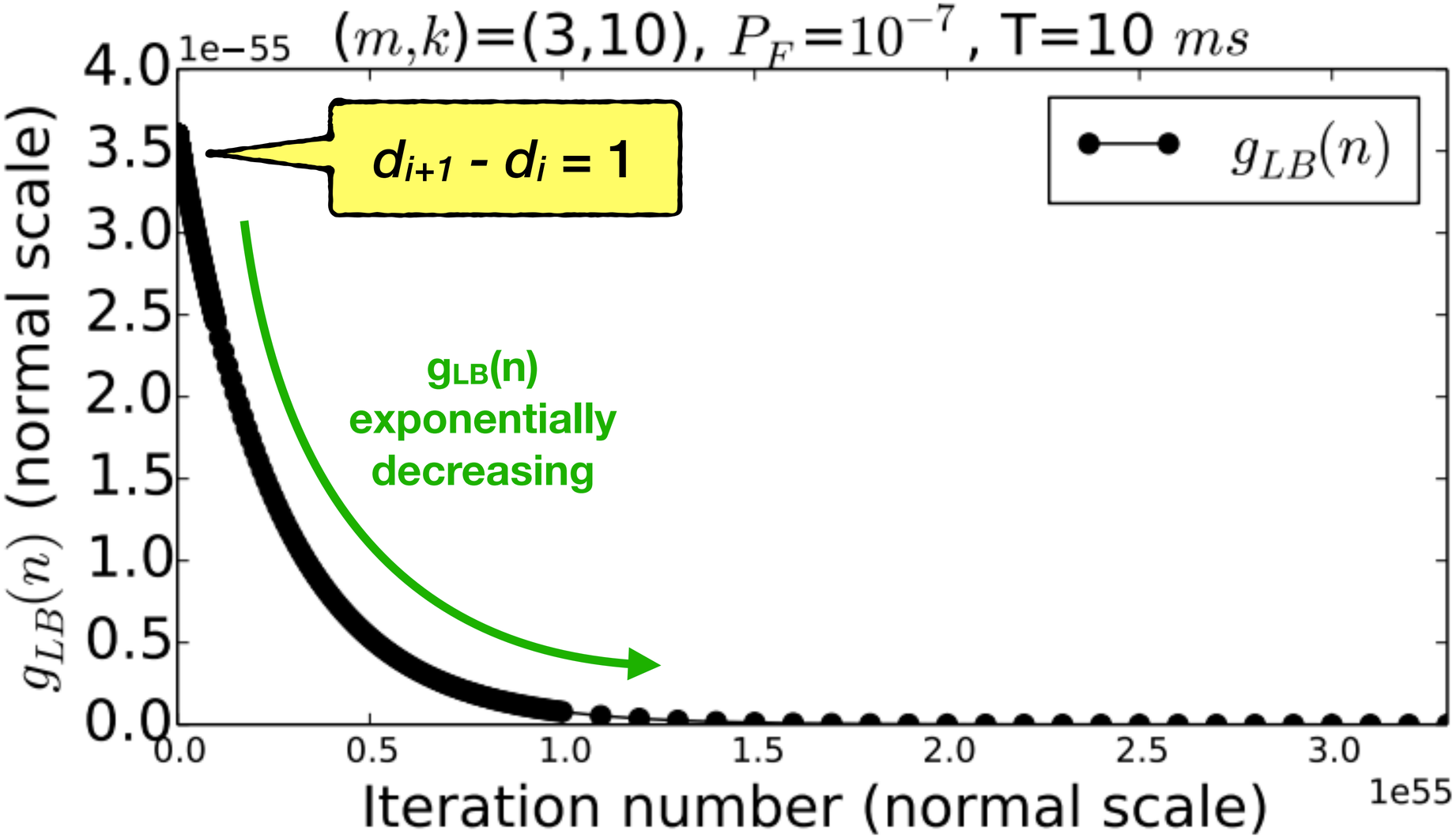

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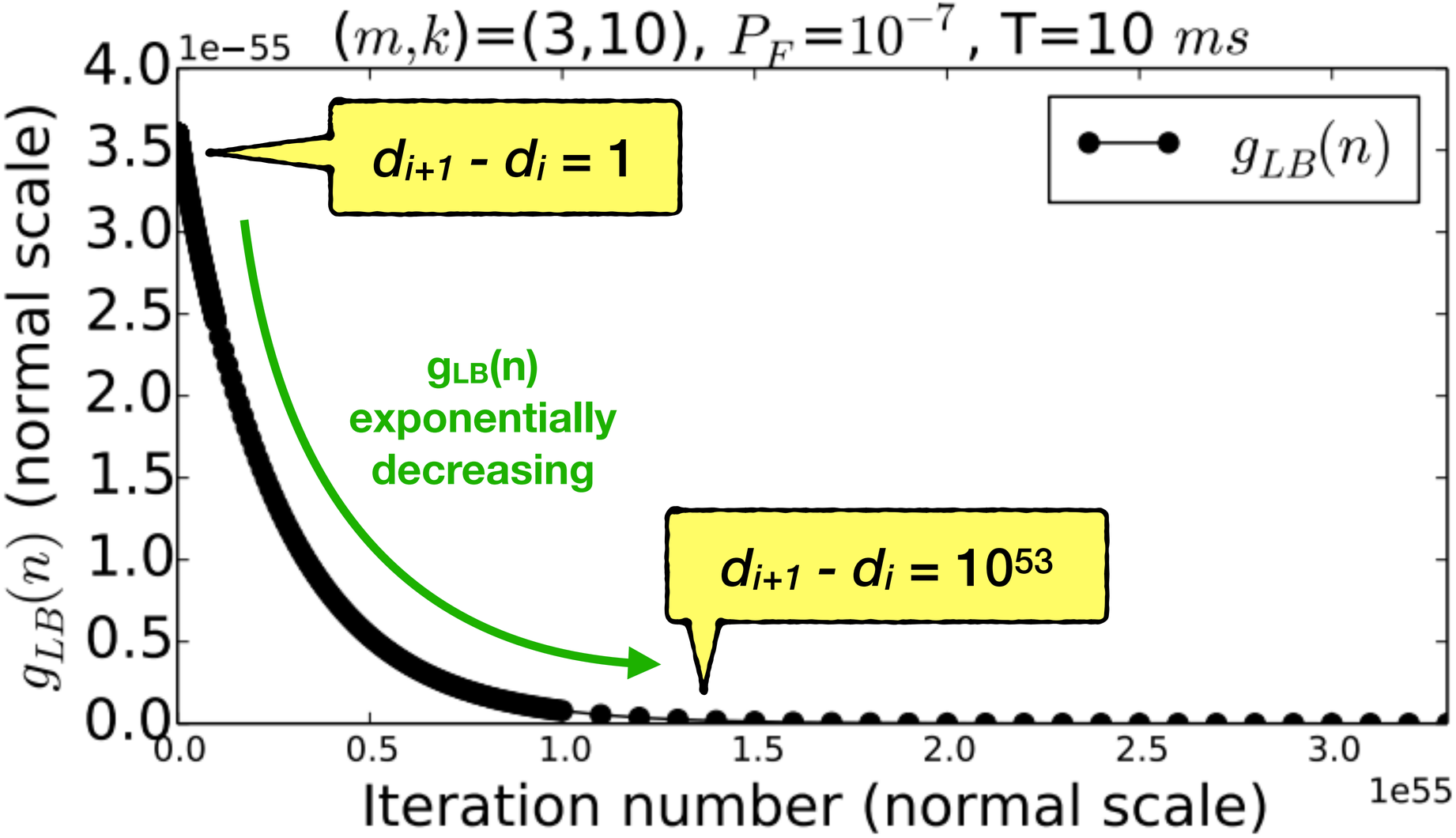

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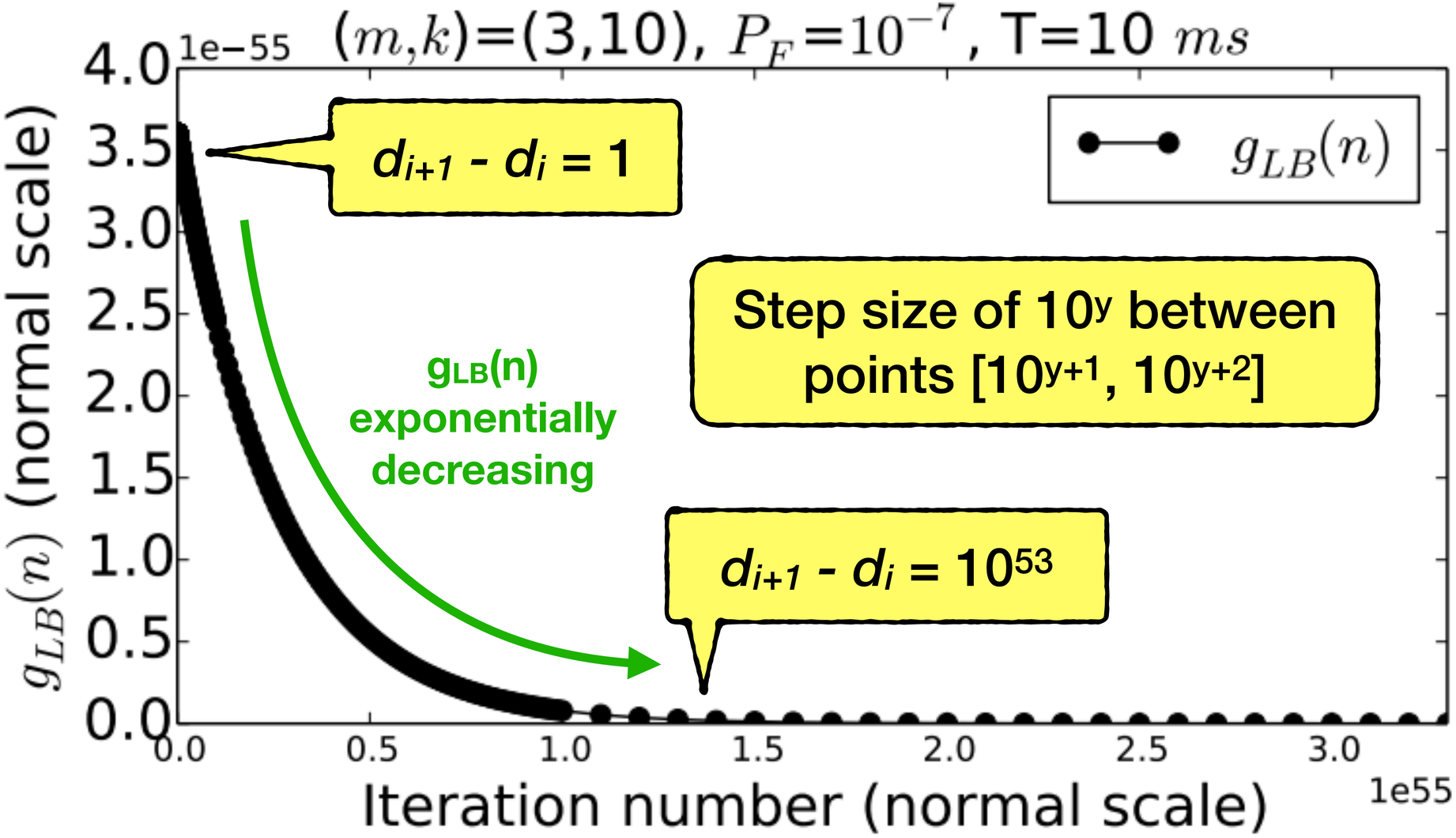

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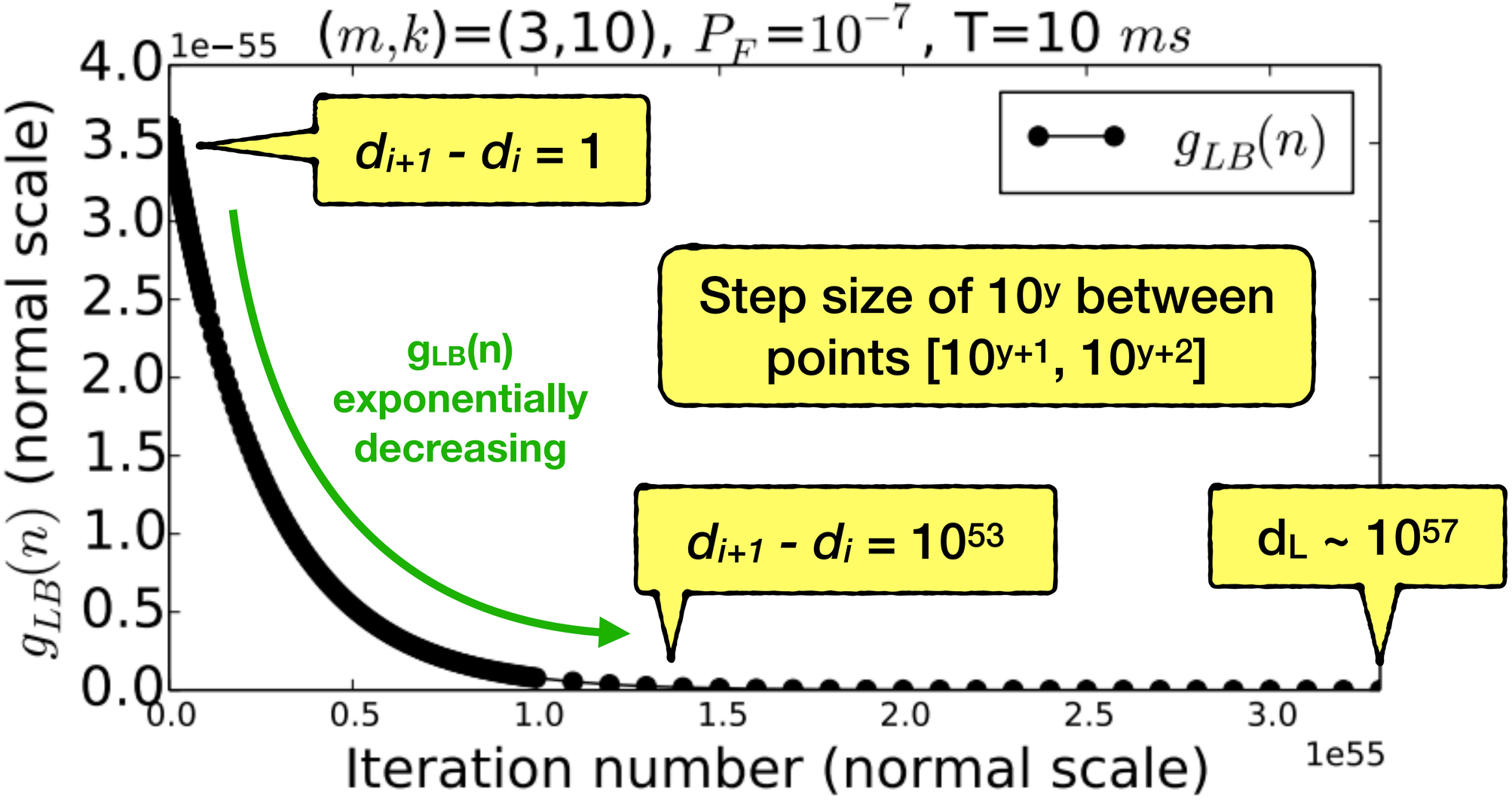

Given F, lower-bound the mean time to failure (MTTF)

Outline

1. Discrete probability density function (dPDF)
   g(n) = P(first (m,k) violation in the n^{th} iteration)
2. Probability density function (PDF)
   f(t) = P(first (m,k) violation at time t)
3. Mean time to failure (MTTF)
   MTTF = E [system lifetime] = $\int_0^\infty t f(t)\, dt$
4. Evaluation

Failure = Violation of the (m,k) constraint:

At least ***m*** iterations, out of any ***k*** consecutive loop iterations must be correct

Approximating MTTF using simulation

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Biased-coin toss experiment

Tails with probability *F*

- system iteration is incorrect

Heads with probability *1 - F*

- system iteration is correct

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Each trial

Repeat coin toss until the (m,k) constraint is violated

Approximating MTTF using simulation

Biased-coin toss experiment

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Heads with probability *1 - F*
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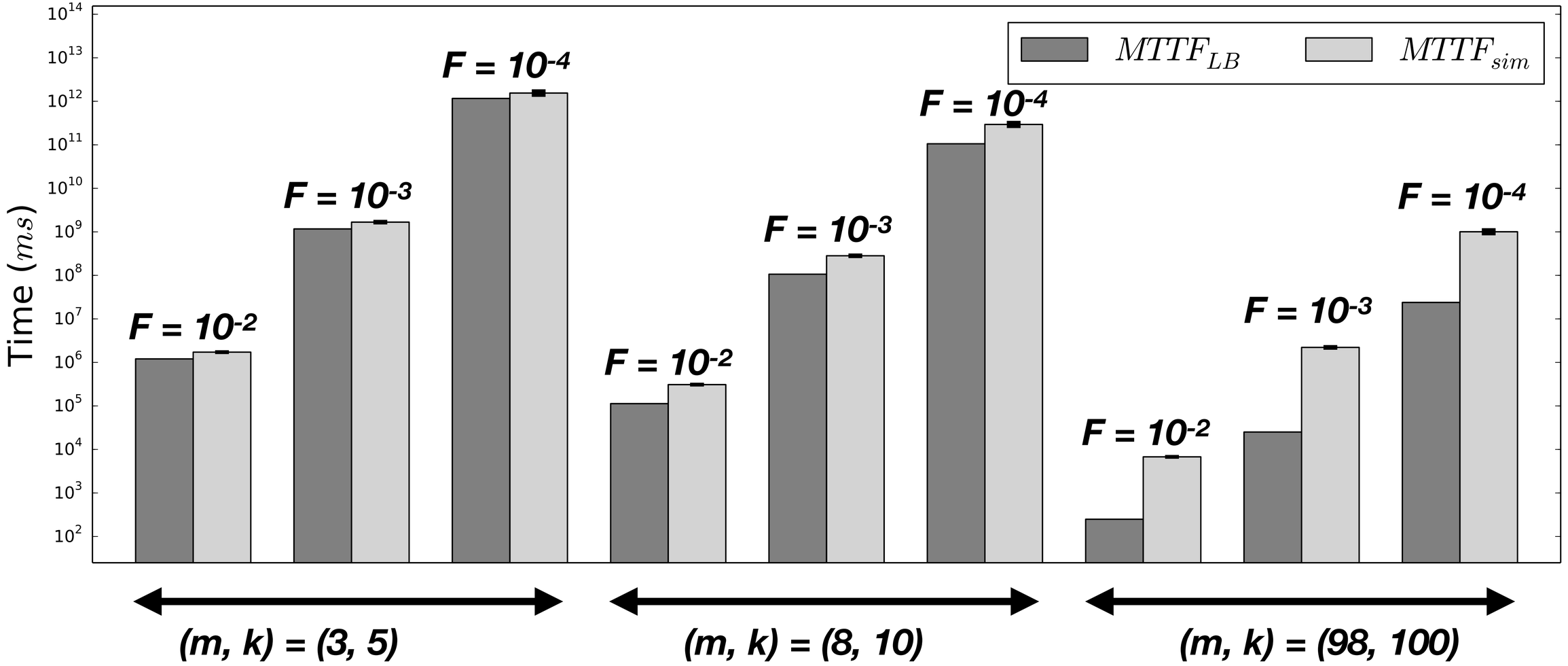
Each trial

Repeat coin toss until the (m,k) constraint is violated

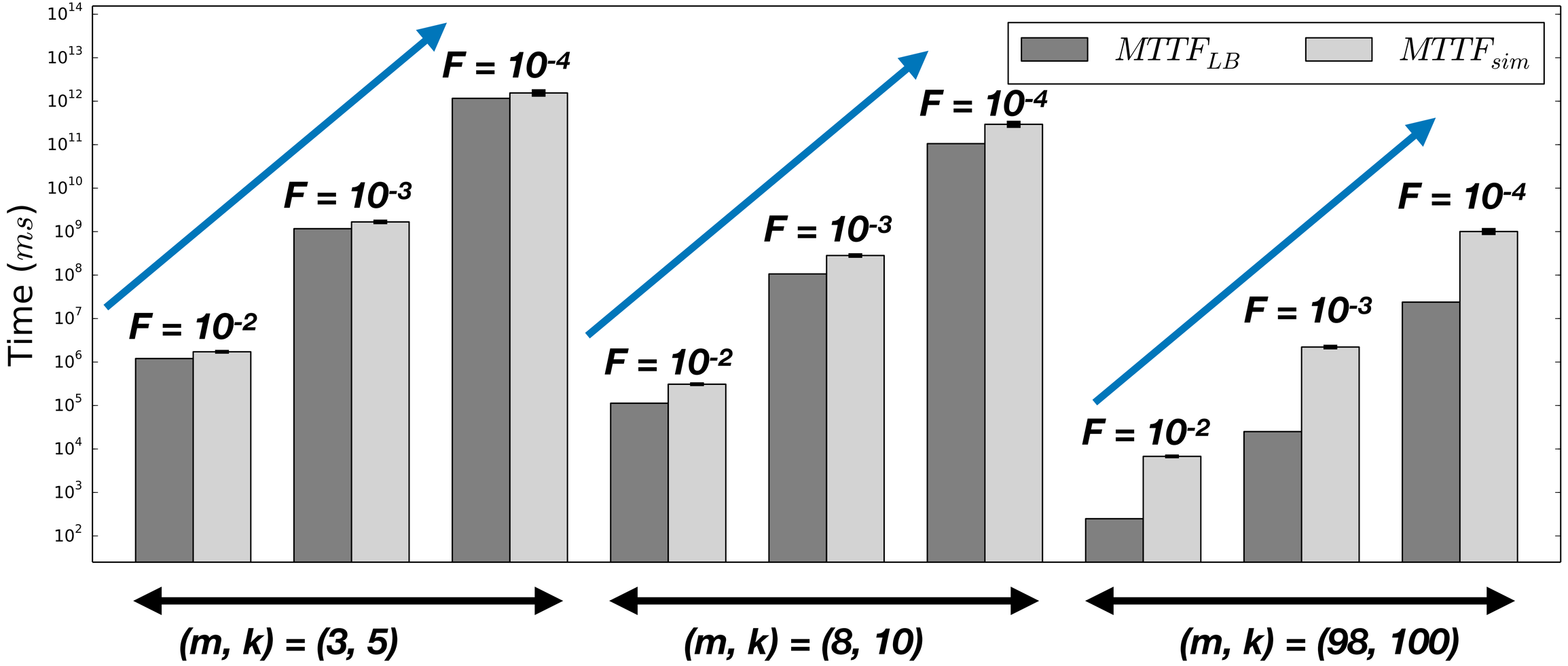
***MTTF*$_{sim}$ = Average tosses per trial x control period**

Comparing $MTTF_{LB}$ and $MTTF_{sim}$



Comparing $MTTF_{LB}$ and $MTTF_{sim}$

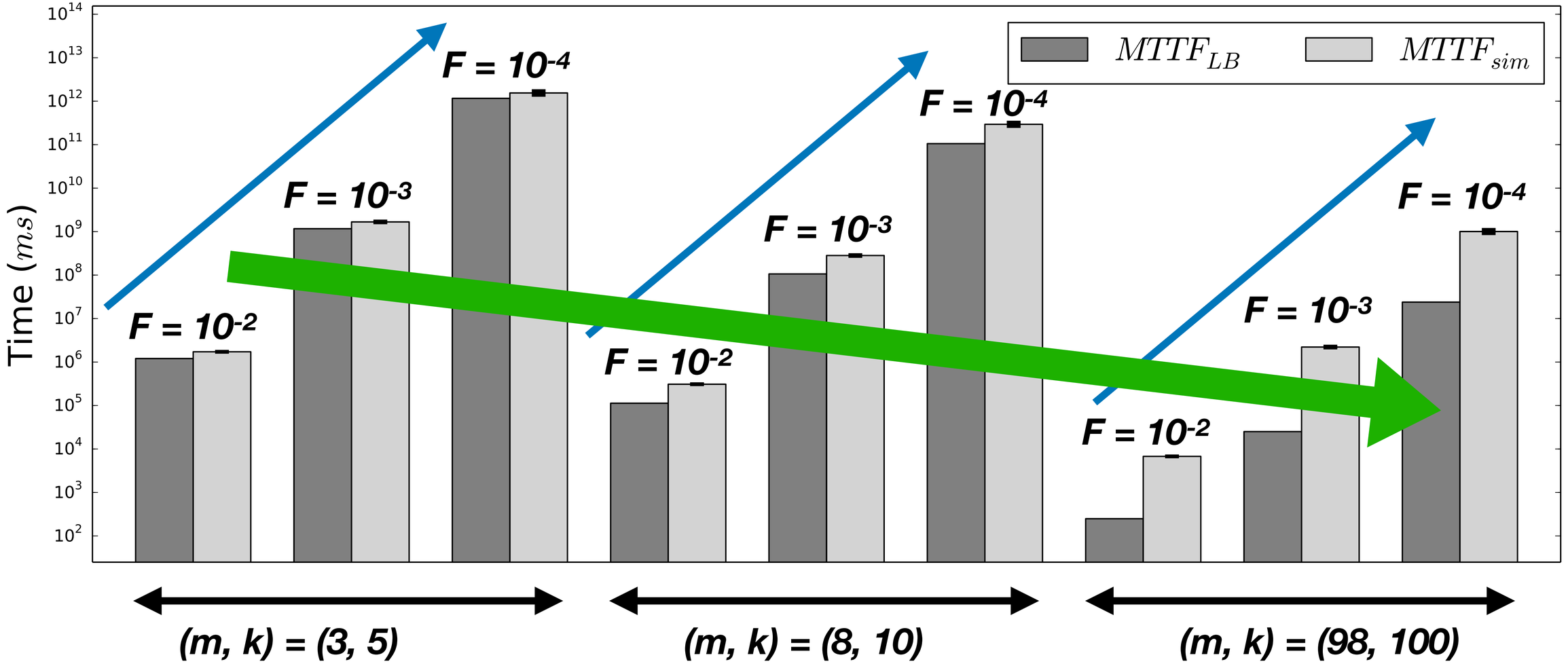
MTTF increases when *F* decreased from 10^{-2} to 10^{-4}



Comparing $MTTF_{LB}$ and $MTTF_{sim}$

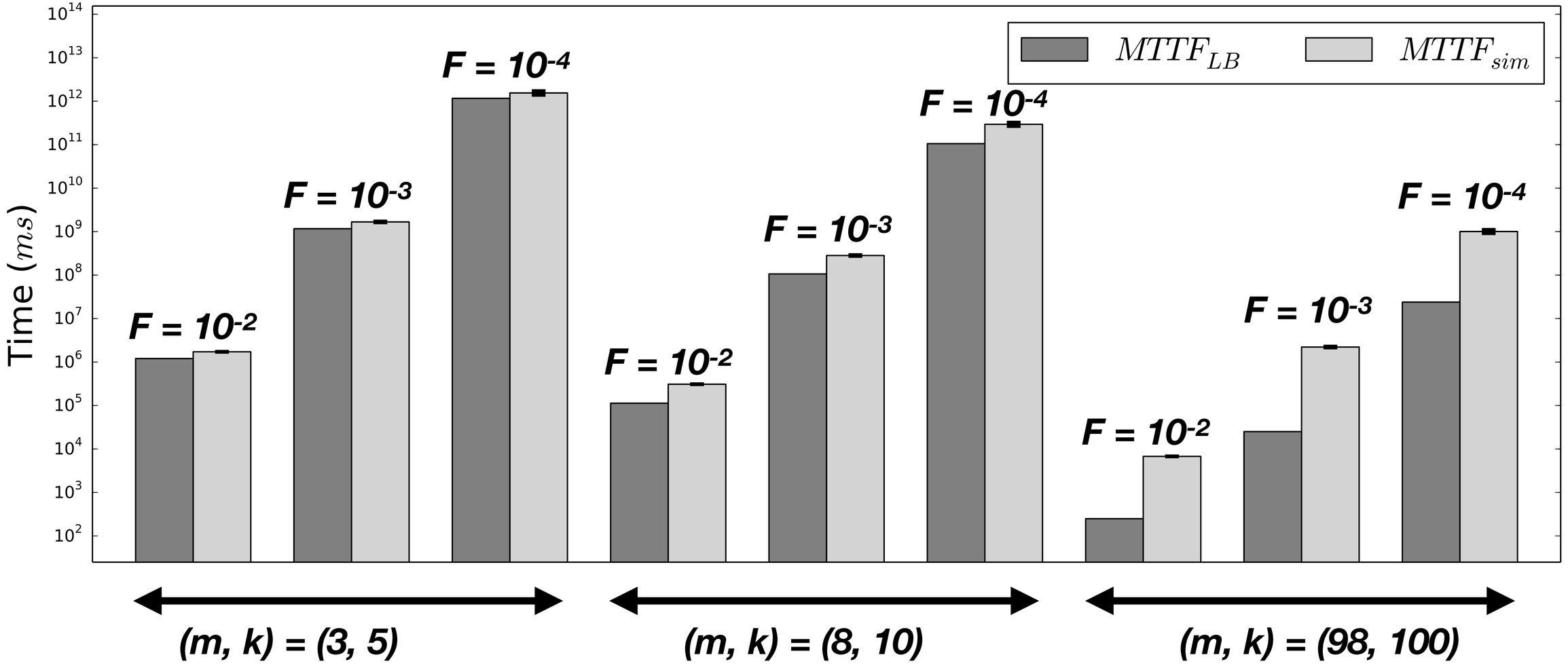
MTTF increases when *F* decreased from 10^{-2} to 10^{-4}

MTTF decreases when *m/k* increased from 3/5 to 98/100



Comparing $MTTF_{LB}$ and $MTTF_{sim}$

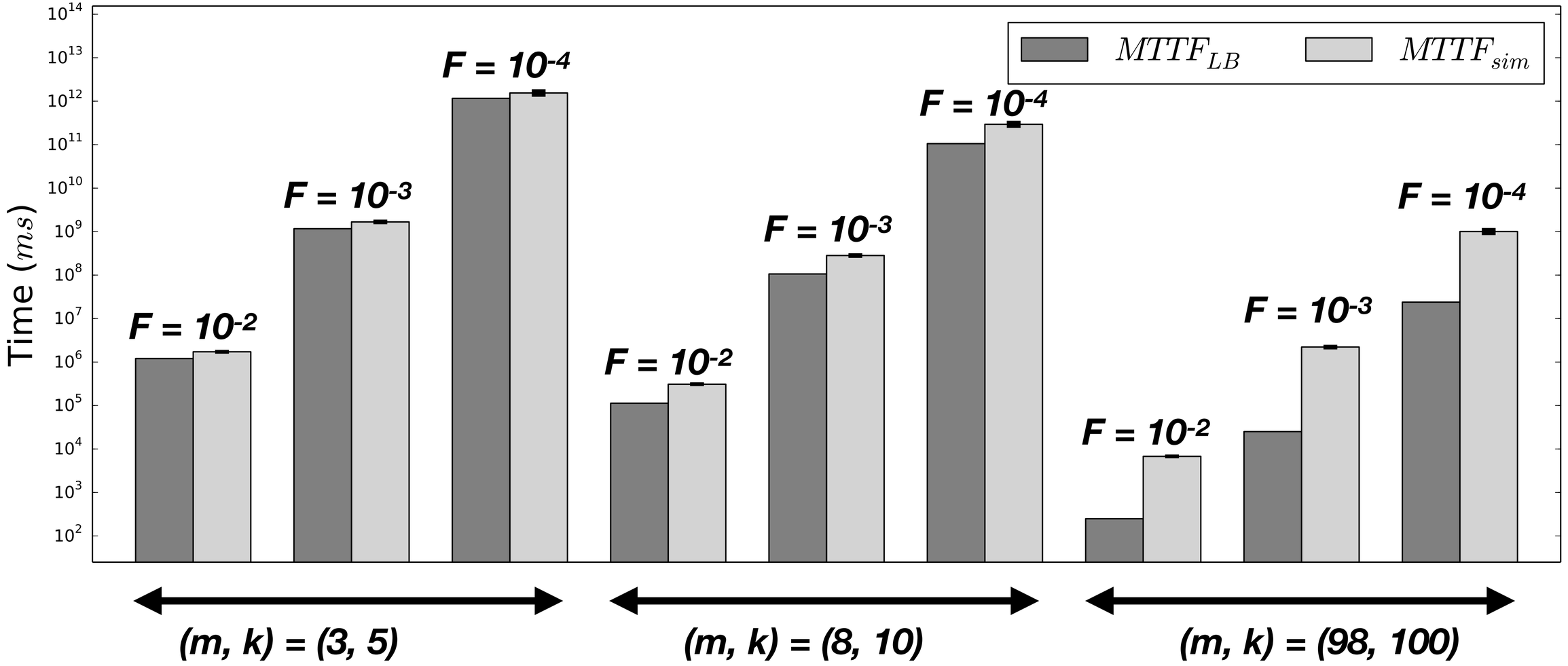
$MTTF_{LB}$ is always less than $MTTF_{sim}$



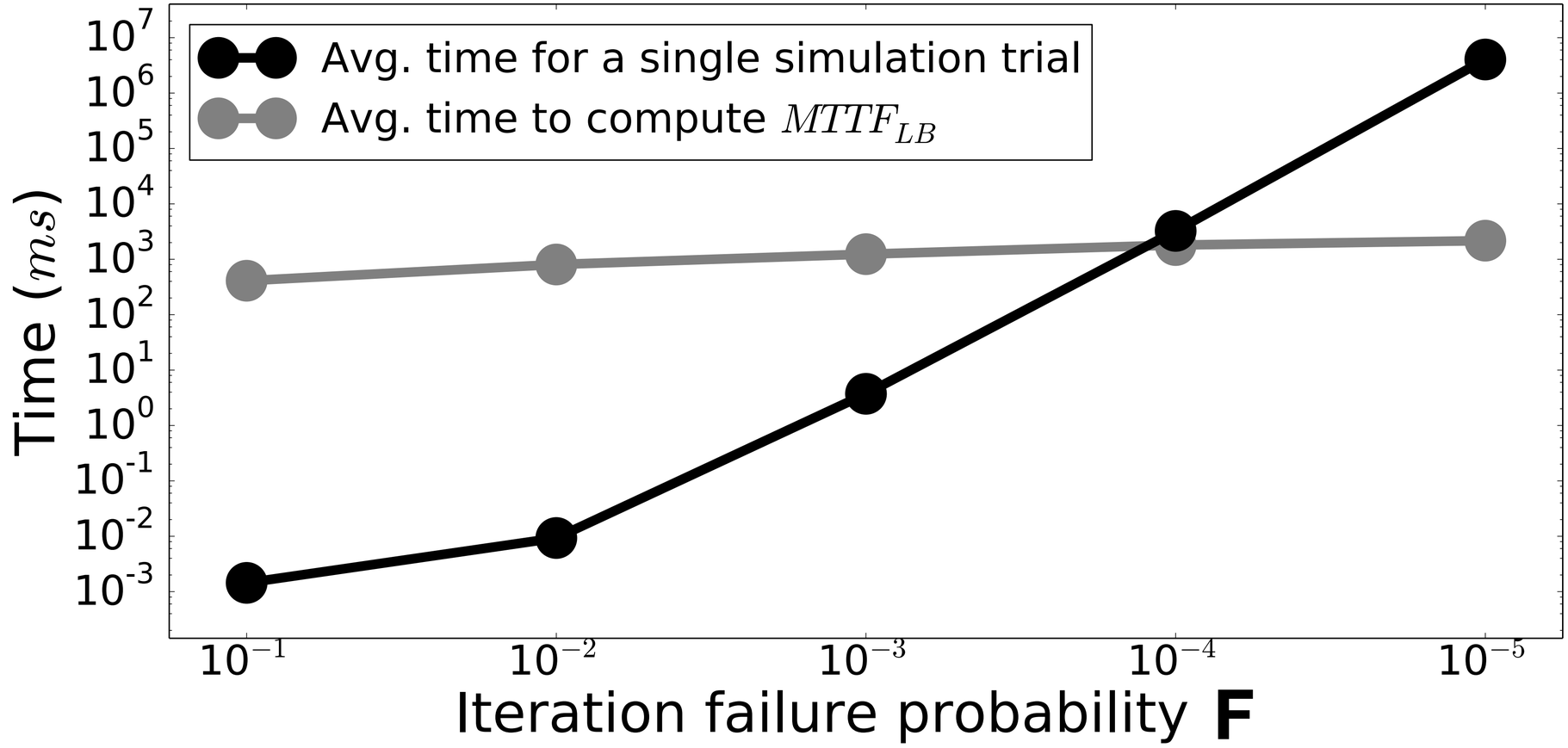
Comparing $MTTF_{LB}$ and $MTTF_{sim}$

$MTTF_{LB}$ is always less than $MTTF_{sim}$

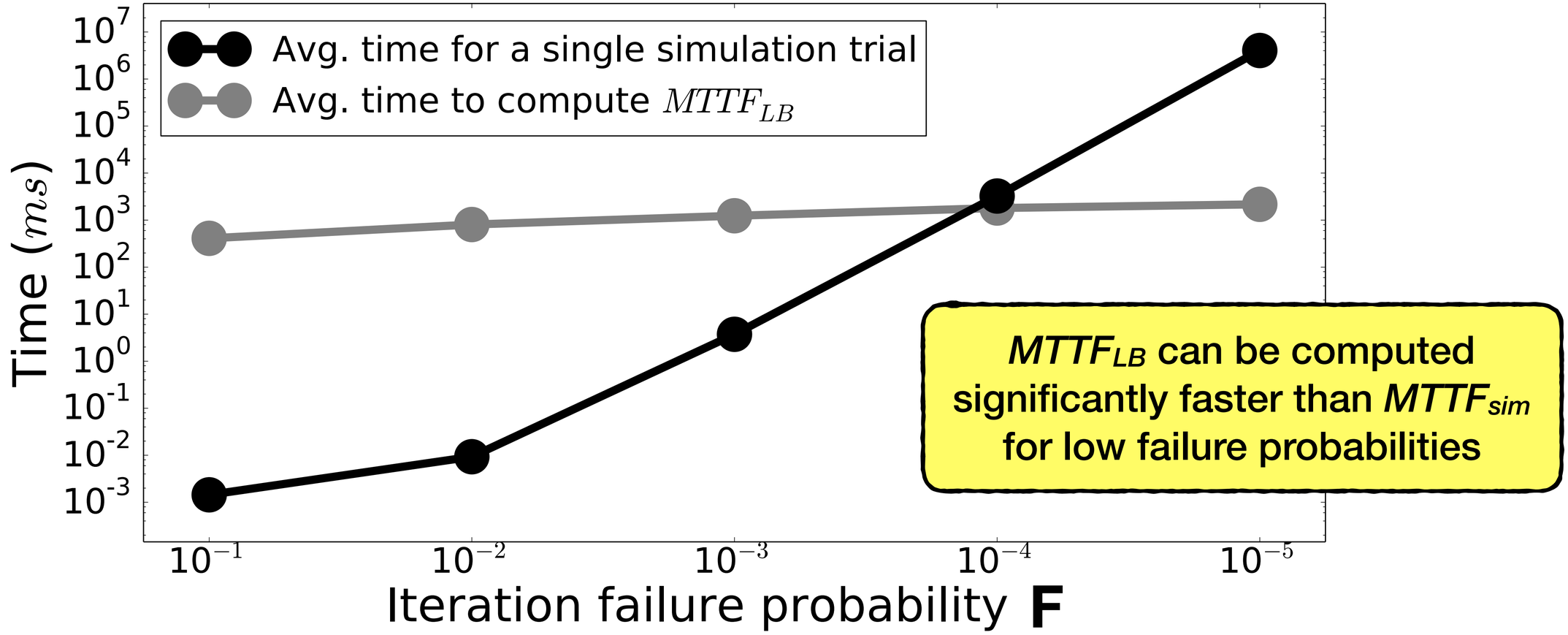
In all cases, $MTTF_{LB}$ and $MTTF_{sim}$ are roughly of the same orders of magnitude



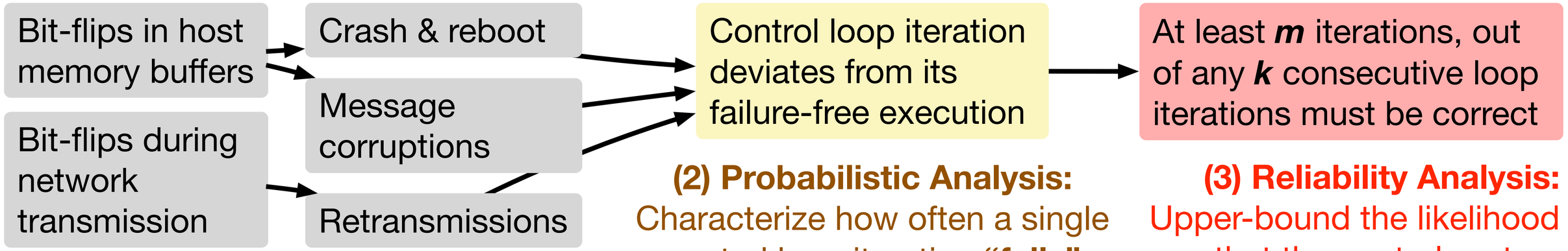
Comparing time to compute $MTTF_{LB}$ and $MTTF_{sim}$



Comparing time to compute $MTTF_{LB}$ and $MTTF_{sim}$



Summary

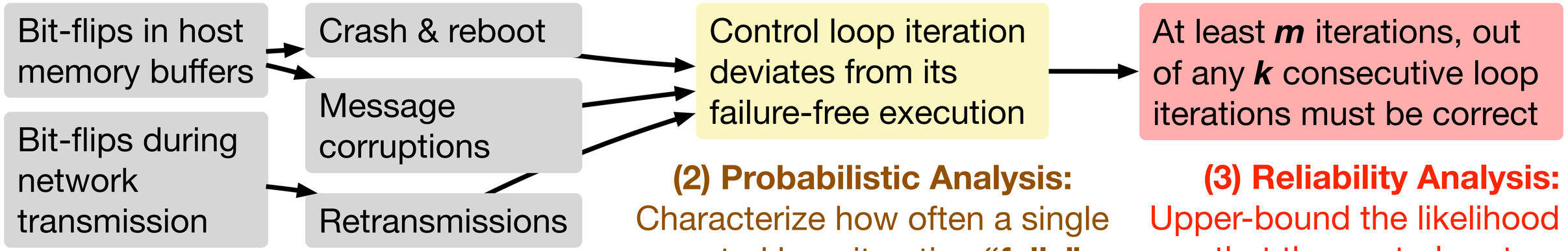


(1) Fault Modeling: Transient faults modeled using **Poisson** distribution, empirically-derived **peak EMI** rates

(2) Probabilistic Analysis: Characterize how often a single control loop iteration **"fails"**

(3) Reliability Analysis: Upper-bound the likelihood that the control system **"fails beyond recovery"**

Summary



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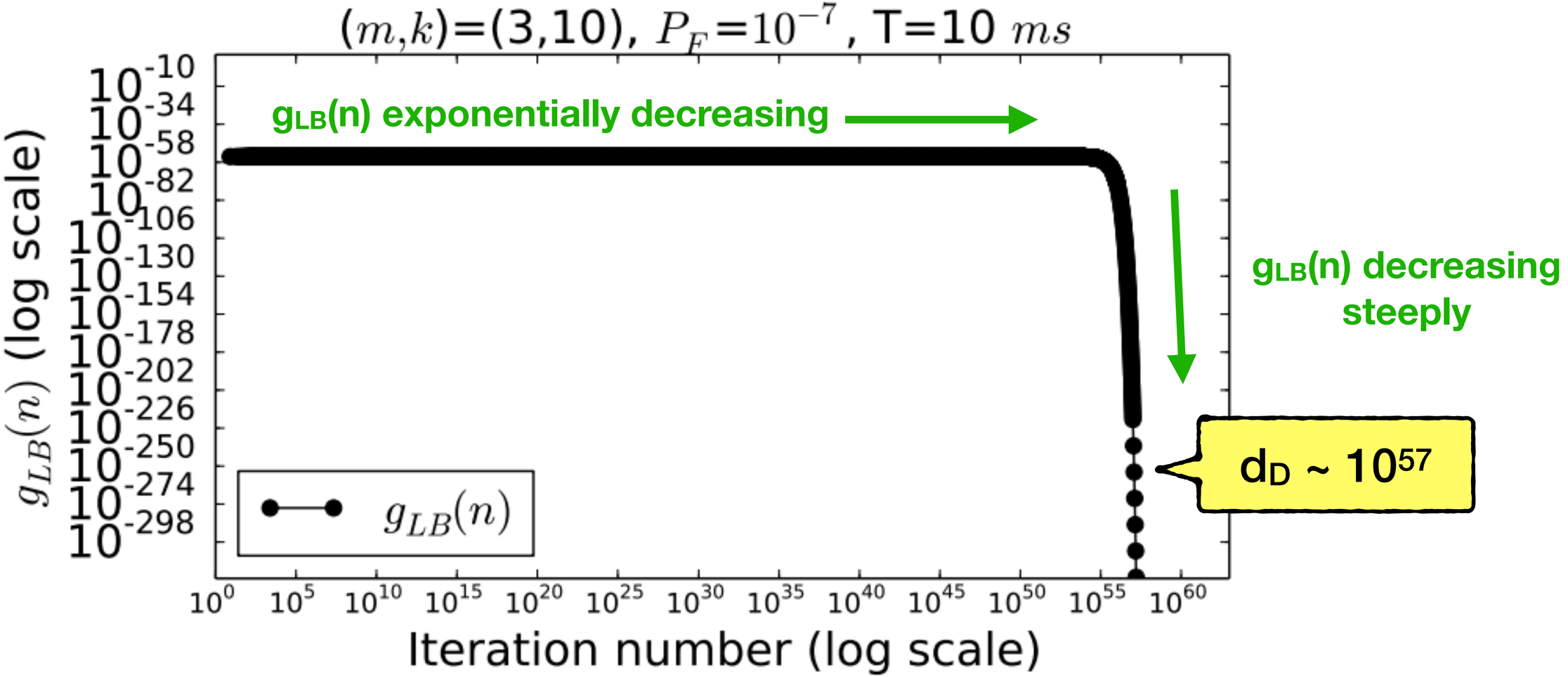
Given a bound *F* on the iteration failure probability, also satisfying the IID property

→ **Safe lower bound on the system MTTF for systems with *(m, k)* constraints**

Thank you. Questions?

Backup


(m,k)=(3,10), P_F=10^{-7}, T=10 ms
$g_{LB}(n)$ (log scale)
10^{-10}
10^{-34}
10^{-58}
10^{-82}
10^{-106}
10^{-130}
10^{-154}
10^{-178}
10^{-202}
10^{-226}
10^{-250}
10^{-274}
10^{-298}
$g_{LB}(n)$
10^{0} 10^{5} 10^{10} 10^{15} 10^{20} 10^{25} 10^{30} 10^{35} 10^{40} 10^{45} 10^{50} 10^{55} 10^{60}
Iteration number (log scale)
$g_{LB}(n)$ exponentially decreasing
$g_{LB}(n)$ decreasing steeply
$d_D \sim 10^{57}$